or $(18 - x_3)/12 = 0$ or $x_3 = 18$ and hence $h = 18$ (f shows its local maximum)

Finally, we get the maximizing dimensions as $l = b = 22.5$ cm and $h = 18$ cm

and $$f_{max} = (22.5)\,(22.5)\,(18) = 9112.5 \text{ cc}$$

Approach of penalty function method

In this method, the basic optimization problem is transformed into an alternative formulation so that numerical solutions are sought by using sequential unconstrained minimization techniques (SUMTs or SUM techniques). The approach is presented as follows with a general optimization problem.

Find X that minimizes $f(X)$

subject to $g_j(X) \le 0$, $j = 1, 2, ..., m$, where $X = [X_i]$, $i = 1, 2, 3, ..., n$

This problem can be converted into an unconstrained minimization problem by defining a function,

$$\phi_k = \phi(X, r_k) = f(X) + r_k \sum_{j=1}^{m} G_j[g_j(X)]$$

where G_j is some function (such as reciprocal) of constraint g_j, and r_k is a positive constant called penalty parameter and $\left[r_k \sum_{j=1}^{m} G_j[g_j(X)\right]$ is called the penalty term. Its significance is explained in the sections to follow.

Now if the unconstrained minimization of function ϕ_k is repeated for a sequence of values of penalty parameters $r_1, r_2, ..., r_k$, the solution can be converged to the original problem. Thus, we are using the penalty function in a sequence of unconstrained minimization problem. Therefore, the method is also called *sequential unconstrained minimization technique* or *penalty function method.* This method can be carried out in three different ways, which are named *exterior*, *interior* and *mixed penalty* function methods as discussed in the sections that follow.

11.6 The Penalty Function

In fact, the penalty terms differ in the way the penalty is assigned. We can use the penalty functions in three ways as follows.

1. Exterior penalty function
2. Interior penalty function
3. Mixed penalty function

The first type of penalty method uses the penalty function to penalize the infeasible points but not the feasible points. In such methods, every sequence of unconstrained optimization attains an improved yet infeasible solution. These methods are known as *exterior penalty methods.*

Working Rule: The Procedural Steps

Step 1: Start at any point X_1 and a suitable value of r_1 (i.e. set $k = 1$).

Step 2: Find the vector X_k^* that minimizes $\phi(X, r_k) = f(X) + r_k \sum_{j=1}^{m} \langle g_j(X)\rangle^q$.

Step 3: Check if the point X_k^* satisfies all the constraints. If yes, terminate; else go to the next step.

Step 4: Set $k = k + 1$ and select the next value of the penalty parameter satisfying the relation $r_{k+1} > r_k$ (Generally r_{k+1} is taken as $r_{k+1} + cr_k$ where $c > 1$, a constant.)

Step 5: Repeat the process till the function gets the optimal value.

Illustration 11.3 Use the exterior penalty function method to

$$\text{minimize } f = 9x_1^2 + 4x_2^2 + 3x_1 + 3x_2$$

subject to $g(x_1) = 5 - 2x_1 \le 0$ and $g(x_2) = 2x_2 - 3 \ge 0$. Also show the convergence of the function under the penalty r.

Solution We shall rewrite the problem as

$$\text{minimize } f = 9x_1^2 + 4x_2^2 + 3x_1 + 3x_2$$

subject to $g(x_1) = \left(\frac{5}{2}\right) - x_1 \le 0$ and $g(x_2) = \left(\frac{3}{2}\right) - x_2 \le 0.$

Solving the problem by the differential calculus method of unconstrained minimization [initial feasible point x_1 is not required (q is chosen as 2)].

$$\phi(X_1, r) = 9x_1^2 + 4x_2^2 + 3x_1 + 3x_2 + r\left[\max\left\{0, \frac{5}{2} - x_1\right\}\right]^2 + r\left[\max\left\{0, \frac{3}{2} - x_2\right\}\right]^2 \quad \text{(i)}$$

where r is the penalty of the function.

The necessary conditions for a multivariable unconstrained problem are written by calculating the partial derivatives and equating them to zero, i.e.

$$\frac{\partial \phi}{\partial x_1} = 18x_1 + 3 - 2r\left[\max\left\{0, \frac{5}{2} - x_1\right\}\right] \qquad \text{(ii)}$$

and

$$\frac{\partial \phi}{\partial x_2} = 8x_2 + 3 - 2r\left[\max\left\{0, \frac{3}{2} - x_2\right\}\right] \qquad \text{(iii)}$$

These equations can be written as follows:

From (ii), $\quad$ minimize $[18x_1 + 3,\ 18x_1 + 3 - 2r(2.5 - x_1)]$ $\qquad$ (iv)

and from (iii), $\quad$ minimize $[8x_2 + 3,\ 8x_2 + 3 - 2r(1.5 - x_2)]$ $\qquad$ (v)

Now from (iv), if $18x_1 + 3 = 0 \Rightarrow x_1 = -3/18$.

This violates the first constraint $5 - 2x_1 \le 0$; so $x_1 = -3/18$ cannot be a solution.

And if $\quad 18x_1 + 3 - 2r(2.5 - x_1) = 0$

i.e. $\quad (18 + 2r)x_1 + (3 - 5r) = 0$

then

$$x_1 = \frac{5r - 3}{2r + 18} \qquad \text{(vi)}$$

From (v), we have $8x_2 + 3 = 0 \Rightarrow x_2 = -3/8$.

This violates the second constraint $3 - 2x_2 \le 0$; so, $x_2 = -3/8$ cannot be a solution.

And if $\quad 8x_2 + 3 - 2r(1.5 - x_2) = 0$

i.e. $\quad (8 + 2r)x_2 + (3 - 3r) = 0$

then

$$x_2 = \frac{3r - 3}{2r + 8} \qquad \text{(vii)}$$

Thus, the solution for the above original problem can be obtained from (vi) and (vii) as follows:

$$x_1^* = \lim_{r\to\infty} x_1^*(r) = \lim_{r\to\infty}\left[\frac{5r - 3}{2r + 18}\right] = \frac{5}{2}$$

and

$$x_2^* = \lim_{r\to\infty} x_2^*(r) = \lim_{r\to\infty}\left[\frac{3r - 3}{2r + 8}\right] = \frac{3}{2}$$

and

$$f_{\min} = f^* = f_{\min}(r) = 9\left(\frac{5}{2}\right)^2 + 4\left(\frac{3}{2}\right)^2 + 3\left(\frac{5}{2}\right) + 3\left(\frac{3}{2}\right)$$

$$= 9\left(\frac{25}{4}\right) + 4\left(\frac{9}{4}\right) + 3\left(\frac{5}{2}\right) + 3\left(\frac{3}{2}\right)$$

$$= \left(\frac{225}{4}\right) + \left(\frac{36}{4}\right) + \left(\frac{15}{2}\right) + \left(\frac{9}{2}\right) = \frac{309}{4} = 77.25$$

Hence the optimal solution is: $f_{\min} = 77.25$ at $x_1^* = 5/2$ and $x_2^* = 3/2$.

The convergence of the function can be observed from the values given in Table 11.1.

Table 11.1 Convergence of $f_{\min}(r)$

Value of r	$x_1 = \dfrac{5r-3}{2r+18}$	$x_2 = \dfrac{3r-3}{2r+18}$	$f_{\min}(r)$
0.001	−0.1663704	−0.16648	−0.6386
0.01	−0.16370699	−0.16482	−0.6357
1	0.1	0	0.39
10	1.236842105	0.710526	21.63
100	2.279816514	1.362385	65.129
1000	2.476214073	1.485134	75.891
10000	2.497602158	1.498501	77.112
100000	2.499760022	1.49985	77.236
1000000	2.499976	1.499985	77.249
⋮	⋮	⋮	⋮
∞	2.5 or 5/2	1.5 or 3/2	77.25

Illustration 11.4 Use the exterior penalty function method to

$$\text{maximize } f = -2x_1^2 - 3x_2^2 - x_1 - x_2$$

subject to $g(x_1) = 15 - 3x_1 \le 0$ and $g(x_2) = -x_2 \le 0$.

Solution We shall rewrite the problem as

$$\text{minimize } f = 2x_1^2 + 3x_2^2 + x_1 + x_2$$

subject to $g(x_1) = 5 - x_1 \le 0$ and $g(x_2) = -x_2 \le 0$.

Let us solve the problem by the differential calculus method of unconstrained minimization (q is chosen as 2).

$$\phi(X_1, r) = 2x_1^2 + 3x_2^2 + x_1 + x_2 + r[\max\{0, 5 - x_1\}]^2 + r[\max\{0, -x_2\}]^2 \quad \text{(i)}$$

where r is the penalty of the function.

The necessary conditions for a multivariable unconstrained problem are written by calculating the partial derivatives and equating them to zero.

$$\therefore \quad \frac{\partial \phi}{\partial x_1} = 4x_1 + 1 - 2r[\max\{0, 5 - x_1\}] = 0 \quad \text{(ii)}$$

and

$$\frac{\partial \phi}{\partial x_2} = 6x_2 + 1 - 2r[\max\{0, -x_2\}] = 0 \quad \text{(iii)}$$

These equations can be written as follows:

From (ii), $\quad$ minimize $[4x_1 + 1,\ 4x_1 + 1 - 2r(5 - x_1)]$ (iv)

and from (iii), $\quad$ minimize $[6x_2 + 1,\ 6x_2 + 1 - 2r(-x_2)]$ (v)

Now from (iv), if $4x_1 + 1 = 0 \Rightarrow x_1 = -1/4$.

This contradicts the first constraint $5 - x_1 \le 0$; so $x_1 = -1/4$ cannot be a solution.

And if $\quad 4x_1 + 1 - 2r(5 - x_1) = 0$

i.e. $\quad (4 + 2r)x_1 + (1 - 10r) = 0$

then
$$x_1 = \frac{10r-1}{2r+4} \tag{vi}$$

From (v), we have $6x_2 + 1 = 0 \Rightarrow x_2 = -1/6$.
This violates the second constraint $-x_2 \leq 0$; so, $x_2 = -1/6$ cannot be a solution.

And if
$$6x_2 + 1 - 2r(-x_2) = 0$$
i.e.
$$(6 + 2r)x_2 + 1 = 0$$
then
$$x_2 = \frac{-1}{2r+6} \tag{vii}$$

Thus, the solution for the above original problem can be obtained from (vi) and (vii) as follows:

$$x_1^* = \lim_{r\to\infty} x_1^*(r) = \lim_{r\to\infty}\left[\frac{10r-1}{2r+4}\right] = 5$$

and
$$x_2^* = \lim_{r\to\infty} x_2^*(r) = \lim_{r\to\infty}\left[\frac{-1}{2r+6}\right] = \frac{0}{2} = 0$$

and
$$f_{\min} = f^* = f_{\min}(r) = 2(5)^2 + 4(0)^2 + (5) + (0)$$
$$= 50 + 0 + 5 + 0 = 55$$

Optimal solution: $f_{\min} = 55$ (or $f_{\max} = -55$) at $x_1^* = 5$ and $x_2^* = 0$.

Illustration 11.5 Use the exterior penalty function method to

$$\text{minimize } f = \frac{1}{3}x_1^3 + x_1^2 + x_1 + x_2 + \frac{1}{3}$$

subject to $g(x_1) = 1 - x_1 \leq 0$ and $g(x_2) = x_2 \geq 0$.

Calculate numerically at least 5 steps in convergence of the penalty, r, of the function starting from 0.001 and reaching infinity.

Solution We shall rewrite the problem as

$$\text{minimize } f = \frac{1}{3}x_1^3 + x_1^2 + x_1 + x_2 + \frac{1}{3}$$

subject to $g(x_1) = 1 - x_1 \leq 0$ and $g(x_2) = -x_2 \leq 0$.

We can use the differential calculus method of unconstrained minimization as this method does not require the initial base point of X and is simpler. Further, q is chosen as 2.

$$\phi(X_1, r) = \frac{1}{3}x_1^3 + x_1^2 + x_1 + x_2 + \frac{1}{3} + r[\max(0, 1 - x_1)]^2 + r[\max(0, -x_2)]^2 \tag{i}$$

where r is the penalty of the function.

The necessary conditions for a multivariable unconstrained problem are written by calculating the partial derivatives and equating them to zero.

$$\therefore \quad \frac{\partial \phi}{\partial x_1} = x_1^2 + 2x_1 + 1 - 2r[\max(0, 1 - x_1)] = 0 \tag{ii}$$

and
$$\frac{\partial \phi}{\partial x_2} = 1 - 2r[\max(0, -x_2)] = 0 \quad \text{(iii)}$$

These equations can be written as follows:

From (ii), $\quad \text{minimize}[x_1^2 + 2x_1 + 1,\ x_1^2 + 2x_1 + 1 - 2r(1 - x_1)]$

and from (iii), $\quad \text{minimize}[1,\ 1 + 2rx_2]$ (iv)

Now, if $x_1^2 + 2x_1 + 1 = 0 \quad \Rightarrow \quad x_1 = -1$ (v)

This contradicts the first constraint, i.e. $1 - x_1 \le 0$ or $x_1 \ge 1$.

And if $x_1^2 + 2x_1 + 1 - 2r(1 - x_1) = 0$ (vi)

i.e. $x_1^2 + (2 + 2r)x_1 + (1 - 2r) = 0$

then
$$x_1 = \frac{-(2+2r) \pm \sqrt{(2+2r)^2 - 4(1-2r)}}{2}$$

or
$$x_1 = \frac{-2(1+r) \pm 2\sqrt{(1+r)^2 - (1-2r)}}{2} = -(1+r) \pm \sqrt{(1+r)^2 - (1-2r)}$$

or
$$x_1 = -(1+r) \pm \sqrt{(r^2 + 4r)} \quad \text{(vii)}$$

Similarly, from (v), the only possible relation $1 + 2rx_2 = 0$ yields $x_2 = -1/2r$

$$\therefore \quad x_1 = -1 - r + r\left(1 + \frac{4}{r}\right)^{1/2} \quad \text{or} \quad -1 - r - r\left(1 + \frac{4}{r}\right)^{1/2} \quad \text{(viii)}$$

and
$$x_2 = -\frac{1}{2r} \quad \text{(ix)}$$

As r increases, the values of x converge to optimal values. Thus, the solution for the problem can be obtained by the following relations:

$$x_1^* = \lim_{r\to\infty} x_1^*(r) = \lim_{r\to\infty}\left[-1 - r + r\left(1 + \frac{4}{r}\right)^{1/2}\right] = 1$$

[The second value of $x_1 = \lim_{r\to\infty}\left[-1 - r - r\left(1 + \frac{4}{r}\right)^{1/2}\right] = -3$ violates first constraint; hence ignored.]

And
$$x_2^* = \lim_{r\to\infty} x_2^*(r) = \lim_{r\to\infty}\left[-\frac{1}{2r}\right] = 0$$

And
$$f_{min} = f^* = \lim_{r\to\infty} \phi_{min}(r) = f_{min}(r) = \frac{1}{3} + 1 + 1 + 0 + \frac{1}{3} = \frac{8}{3} = 2.66667$$

The convergence of the function as the penalty, r, values increase from 0.001 to infinity is shown in Table 11.2.

Table 11.2 Convergence of the Function with the Increasing Values of the Penalty

Value of r	$x_1^* = -1 - r + r\left(1 + \frac{4}{r}\right)^{1/2}$	$x_2^*(r) = -\frac{1}{2r}$	$f_{min}(r)$	$\phi_{min}(r)$
0.001	−0.9377	−500	−249.997	−500
0.01	−0.8097	−50	−24.965	−49.9977
1	0.2360	−5	−2.234	−4.9474
10	0.8322	−0.05	2.307	2.001
100	0.9804	−0.005	2.625	2.5840
1000	0.9980	−0.0005	2.662	2.6582
⋮	⋮	⋮	⋮	⋮
∞	1	0	2.6667	2.6667 Optimal

Interior Penalty Function Method

This method is also known as the *barrier method* since the constraint boundaries act as barriers during the minimize process. The interior penalty function was first defined by Carroll. The interactive process is summarized as follows:

Working rule: the procedural steps

1. Start at an initial feasible point X_1 that can satisfy all the constraints with strict inequality sign, i.e. $g_j(X_1) < 0$ for $j = 1, 2, ..., m$, and an initial value of $r_1 > 0$ (i.e. set $k = 1$).
2. Minimize $\phi(X, r_k)$ by using any unconstrained minimization technique and obtain the solution for x_k^*.
3. Test if x_k^* is the optimum solution of the original problem. If yes, stop. Otherwise go to the next step.
4. Find the value of the next penalty parameter, i.e. r_{k+1} as $r_{k+1} = cr_k$ where $c < 1$.
5. Set the new value of $k = k + 1$ and take the new starting point as $x_1 = x_k^*$ and repeat the process.

Limitations

Note: Some points are noteworthy where care should be taken while implementing this process.

1. The starting feasible point (i.e. X_1) may not be readily available in some cases.
2. A suitable value of the initial penalty parameter, r_1, has to be found.
3. A suitable value has to be selected for the multiplication factor, c.
4. Appropriate convergence criteria have to be selected to identify the optimum point.
5. The constraints have to be normalized such that each one of them varies between −1 and 0 only.

Illustration 11.6 Use the interior penalty function method to

$$\text{minimize } f = 9x_1^2 + 4x_2^2 + 3x_1 + 3x_2$$

subject to $g(x_1) = 5 - 2x_1 \le 0$ and $g(x_2) = 2x_2 - 3 \ge 0$. Also show the convergence of the function under the penalty r.

Solution We shall rewrite the problem as

$$\text{minimize } f = 9x_1^2 + 4x_2^2 + 3x_1 + 3x_2$$

subject to $g(x_1) = (5/2) - x_1 \le 0$ and $g(x_2) = (3/2) - x_2 \le 0$.

It is convenient to use the calculus method in this case and so the initial feasible point X_1 is not needed.

Let us first define the ϕ function as

$$\phi(X_1, r) = 9x_1^2 + 4x_2^2 + 3x_1 + 3x_2 - r\left[\frac{1}{\frac{5}{2} - x_1} + \frac{1}{\frac{3}{2} - x_2}\right] \quad \text{(i)}$$

where r is the interior penalty of the function.

The necessary conditions for a multivariable unconstrained problem are written by calculating the partial derivatives and equating them to zero.

i.e.
$$\frac{\partial \phi}{\partial x_1} = 18x_1 + 3 - r\left[\frac{1}{(2.5 - x_1)^2}\right] = 0 \quad \text{(ii)}$$

and
$$\frac{\partial \phi}{\partial x_2} = 8x_2 + 3 - r\left[\frac{1}{(1.5 - x_2)^2}\right] = 0 \quad \text{(iii)}$$

Now from (ii), $18x_1 + 3 = r\left[\frac{1}{(2.5 - x_1)^2}\right]$ or $3(6x_1 + 1)(2.5 - x_1)^2 = r$

Thus, intuitively we can find the convergence of $x_1 \to 2.5$ or $-1/6$ as the limit tends to zero ($r \to 0$).

The negative answer ($-1/6$) violates the first constraint $5 - 2x_1 \le 0$.

So, we consider $x_1^* = 2.5$ or $5/2$.

Now from (iii),

$$8x_2 + 3 = r\left[\frac{1}{(1.5 - x_2)^2}\right] \quad \text{or} \quad (8x_2 + 3)(1.5 - x_2)^2 = r$$

Thus, the convergence can be noticed at $x_2 \to -3/8$ or $3/2$ as the limit r tends to zero ($r \to 0$).

The negative answer ($-3/8$) violates the second constraint $3 - 2x_2 \le 0$.

So, we consider $x_2^* = 1.5$ or $3/2$.

And
$$f_{\min} = f^* = f_{\min}(r) = 9\left(\frac{5}{2}\right)^2 + 4\left(\frac{3}{2}\right)^2 + 3\left(\frac{5}{2}\right) + 3\left(\frac{3}{2}\right)$$

$$= 9\left(\frac{25}{4}\right) + 4\left(\frac{9}{4}\right) + 3\left(\frac{5}{2}\right) + 3\left(\frac{3}{2}\right)$$

$$= \left(\frac{225}{4}\right) + \left(\frac{36}{4}\right) + \left(\frac{15}{2}\right) + \left(\frac{9}{2}\right) = \frac{309}{4} = 77.25$$

Optimal solution: $f_{\min} = 77.25$ at $x_1^* = 5/2$ and $x_2^* = 3/2$.

The convergence of the function can be observed by calculating the values of x_1^* and x_2^* and f^* by giving the values to r = 1000, 100, 10, 1, 0.1, 0.01, 0.001, ..., and so on till the function consistently converges.

Illustration 11.7 Use the interior penalty function method to

$$\text{maximize } f = -2x_1^2 - 3x_2^2 - x_1 - x_2$$

subject to $g(x_1) = 15 - 3x_1 \leq 0$ and $g(x_2) = -x_2 \leq 0$.

Solution We shall rewrite the problem as

$$\text{minimize } f = 2x_1^2 + 3x_2^2 + x_1 + x_2$$

subject to $g(x_1) = 5 - x_1 \leq 0$ and $g(x_2) = -x_2 \leq 0$.

Let us use the calculus method in this case so that the initial feasible point X_1 is not necessary.

Let us first define the ϕ function as

$$\phi(X_1, r) = 2x_1^2 + 3x_2^2 + x_1 + x_2 - r\left[\frac{1}{5-x_1} + \frac{1}{-x_2}\right] \quad \text{(i)}$$

where r is the interior penalty of the function.

The necessary conditions for a multivariable unconstrained problem are written by calculating the partial derivatives and equating them to zero.

i.e.

$$\frac{\partial \phi}{\partial x_1} = 4x_1 + 1 - r\left[\frac{1}{(5-x_1)^2}\right] = 0 \quad \text{(ii)}$$

and

$$\frac{\partial \phi}{\partial x_2} = 6x_2 + 1 - r\left[\frac{1}{(-x_2)^2}\right] = 0 \quad \text{(iii)}$$

Now from (ii), $4x_1 + 1 = r\left[\frac{1}{(5-x_1)^2}\right]$ or $(4x_1 + 1)(5 - x_1)^2 = r$

Thus, the convergence can be found as the limit tends to zero ($r \to 0$) at $x_1 = 5$ or $-1/4$. The negative answer (–1/4) violates the first constraint $15 - 3x_1 \leq 0$.

So, we consider $x_1^* = 5$.

Now from (iii), $6x_2 + 1 = r\left[\frac{1}{(-x_2)^2}\right]$ or $(6x_2 + 1)(-x_2)^2 = r$

i.e.

$$6x_2^3 + x_2^2 - r = 0$$

As $x_2 \to 0$ or $-1/6$, the penalty r tends to zero ($r \to 0$).

The negative answer (–1/6) violates the second constraint $-x_2 \leq 0$.

So, we consider $x_2^* = 0$

And

$$f_{\min} = f^* = f_{\min}(r) = 2(5)^2 + 4(0)^2 + (5) + (0)$$
$$= 50 + 0 + 5 + 0 = 55$$

Optimal solution: $f_{\min} = 55$ (or $f_{\max} = -55$) at $x_1^* = 5$ and $x_2^* = 0$.

Illustration 11.8 Use the interior penalty function method to

$$\text{minimize } f = \frac{1}{3}x_1^3 + x_1^2 + x_1 + x_2 + \frac{1}{3}$$

subject to the constraints $g(x_1) = 1 - x_1 \le 0$ and $g(x_2) = x_2 \ge 0$. Calculate numerically at least six steps in convergence of the penalty, r, of the function starting from $r = 1000$.

Solution It is convenient to use the calculus method in this case and so the initial feasible point X_1 is not necessary.

Let us first define the ϕ function as

$$\phi(X, r) = \frac{x_1^3}{3} + x_1^2 + x_1 + x_2 + \frac{1}{3} - r\left[\frac{1}{-x_1 + 1} - \frac{1}{x_2}\right]$$

where r is the interior penalty of the function.

Applying the necessary conditions, we have

$$\frac{\partial \phi}{\partial x_1} = x_1^2 + 2x_1 + 1 - \frac{r}{(1 - x_1)^2} = 0 \qquad \text{(i)}$$

and

$$\frac{\partial \phi}{\partial x_2} = 1 - \frac{r}{x_2^2} = 0 \qquad \text{(ii)}$$

Relation (i) can be rewritten as $(x_1 + 1)^2 - r/(x_1 - 1)^2 = 0$

i.e.

$$(x_1^2 - 1)^2 - r = 0 \quad \Rightarrow \quad (x_1^2 - 1)^2 = r$$

i.e.

$$x_1^2 - 1 = \sqrt{r} \quad \Rightarrow \quad x_1^2 = 1 + \sqrt{r}$$

or

$$x_1(r) = \sqrt{1 + \sqrt{r}}.$$

And Eq. (ii) gives $x_2(r) = \sqrt{r}$.

Further, the ϕ function is rewritten as

$$\therefore \quad \phi_{\min}(r) = \frac{1}{3}\left(\sqrt{1+\sqrt{r}}\right)^3 + \left(\sqrt{1+\sqrt{r}}\right)^2 + \left(\sqrt{1+\sqrt{r}}\right) + \sqrt{r} + \frac{1}{3} - \frac{1}{\left(\frac{1}{r}\right) - \sqrt{\left(\frac{1}{r}\right)^{3/2} + \left(\frac{1}{r}\right)^2}}$$

To get the solution of the problem, we use the theory of limits as follows:

$$f_{\min}(r) = \lim_{r \to \infty} \phi_{\min}(r)$$

$$x_1^* = \lim_{r \to \infty} x_1^*(r) \quad \text{and} \quad x_2^* = \lim_{r \to \infty} x_2^*(r)$$

The values of $f_{\min}$, x_1^*, x_2^* corresponding to a decreasing sequence of values of r are shown in Table 11.3.

Table 11.3 Values of $f_{min}(r)$ with Decreasing Values of r

Value of r	$x_1^*(r)=\sqrt{1+\sqrt{r}}$	$x_2^*(r)=\sqrt{r}$	$\phi_{min}(r)$	$f_{min}(r)$
1000	5.7116	31.6228	376.2646	132.4003
100	3.3166	10.0000	89.9772	36.8109
10	2.0402	3.1623	25.3048	12.5286
1	1.4142	1.0000	9.1046	5.6904
0.1	1.1473	0.3162	4.6117	3.6164
0.01	1.0488	0.1000	3.2716	2.9667
0.001	1.0157	0.0316	2.8569	2.7615
0.0001	1.0049	0.0100	2.7267	2.6967
0.00001	1.0016	0.00316	2.6856	2.6762
0.000001	1.0005	0.00100	2.6727	2.6697
⋮	⋮	⋮	⋮	⋮
0	1	0	2.66667 = 8/3	2.66667 = 8/3