or $\frac{1}{p} = \frac{1}{2}\left(\frac{dp}{dy} - \frac{1}{p^2}\frac{dp}{dy}\right)$ $\left(\text{since } \frac{dx}{dy} = \frac{1}{p}\right)$

or $\frac{1}{p} = \frac{1}{2}\left(1 - \frac{1}{p^2}\right)\frac{dp}{dy}$

Separating the variables, we get $dy = \frac{1}{2}\left(p - \frac{1}{p}\right)dp$

Integrating both sides, we get

$$y = \frac{1}{4}p^2 - \frac{1}{2}\ln p + C \qquad (2)$$

Equations (1) and (2) constitute the complete solution of the given differential equation.

## 3.7 CLAIRAUT EQUATION

A differential equation of the form

$$y = px + f(p) \qquad (1)$$

is called **Clairaut equation** named after the French mathematician A.C. Clairaut (1713 – 1765).

This equation is solvable for $y$. To solve it, we differentiate w.r.t. $x$ to obtain

$$p = \frac{dy}{dx} = p + xp' + f'(p)p' = [x + f'(p)]p' + p$$

or $[x + f'(p)]p' = 0$

Rejecting the factor $x + f'(p)$ which does not involve $\frac{dp}{dx}$, we have

$$p' = \frac{dp}{dx} = 0$$

or $p = C = \text{constant}$ (2)

Eliminating $p$ between equations (1) and (2), we get

$$y = Cx + f(c) \qquad (3)$$

which is the required solution of the Clairaut equation.

Thus it appears that to solve Clairaut equation, it is necessary only to replace $p$ by $C$.

**NOTE:** If we eliminate $p$ between $x + f'(p) = 0$ and equation (1), we get a solution which does not contain any arbitrary constant, and hence, is not a particular solution of equation (3). Such a solution is called **singular solution**.

**EXAMPLE (4):** **Solve the following differential equations :**

(i) $y = px + \sqrt{a^2p^2 + b^2}$ (ii) $y = p(x - b) + \frac{a}{p}$

**SOLUTION:** (i) $y = px + \sqrt{a^2p^2 + b^2}$

This equation is in standard form of Clairaut equation. Its solution can be written by replacing $p$ by $C$.

Thus $y = Cx + \sqrt{a^2C^2 + b^2}$ is the solution of the given equation.

(ii) $y = p(x-b)+\frac{a}{p}$

This equation can be written as $y = px - pb + \frac{a}{p}$

which is the general form of Clairaut equation.

Its solution can be written by replacing p by C. Thus

$y = Cx - Cb + \frac{a}{C}$ is the solution of the given equation.

## 3.8 EQUATIONS REDUCIBLE TO CLAIRAUT FORM

The equation of the form

$$y^2 = pxy + f\left(\frac{py}{x}\right) \quad (1)$$

can be reduced to Clairaut form by making the substitutions $u = x^2$, and $v = y^2$.

Put $u = x^2$ then $\frac{du}{dx} = 2x$, $v = y^2$ therefore $\frac{dv}{dx} = 2y\frac{dy}{dx} = 2yp$

Now $\frac{dv}{du} = \frac{dv}{dx}\frac{dx}{du} = \frac{py}{x}$

Then equation (1) becomes

$$v = u\frac{dv}{du} + f\left(\frac{dv}{du}\right)$$

which is obviously the Clairaut equation.

**EXAMPLE (5):** **Solve the differential equation :**

$$x^2y^2 = px^3y + p^2y^2 + pxy + 9x^2.$$

**SOLUTION:** The differential equation can be written as

$$y^2 = pxy + \frac{p^2y^2}{x^2} + \frac{py}{x} + 9 \quad (1)$$

Let $x^2 = u$ and $y^2 = v$, then $2x\,dx = du$ and $2y\,dy = dv$

or $\frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\frac{dy}{dx} = \frac{py}{x}$

Substituting in equation (1), we get

$$v = \frac{x}{y}\left(\frac{dv}{du}\right)xy + \left(\frac{dv}{du}\right)^2 + \frac{dv}{du} + 9$$

$$= x^2\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 + \frac{dv}{du} + 9$$

$$= u\frac{dv}{dx} + \left(\frac{dv}{du}\right)^2 + \frac{dv}{du} + 9 \quad (2)$$

Let $\frac{dv}{dx} = P$, then equation (2) becomes

$$v = up + (P^2 + P + 9)$$

which is of Clairaut form. Thus its general solution is given by

$$v = uC + (C^2 + C + 9)$$

or
$$y^2 = x^2 C + (C^2 + C + 9)$$
$$= C(x^2 + C + 1) + 9$$

## 3.9 EQUATIONS REDUCIBLE TO CLAIRAUT FORM BY TRANSFORMATION

**EXAMPLE (6):** **Solve the differential equation :**

$$y p^2 + x^3 p - x^2 y = 0.$$

**SOLUTION:**
$$y p^2 + x^3 p - x^2 y = 0 \qquad (1)$$

Let $x^2 = u$, and $y^2 = v$, then

$$2x\,dx = du \quad \text{and} \quad 2y\,dy = dv$$

$$\frac{dv}{du} = \frac{y}{x}\frac{dy}{dx} = \frac{yp}{x} = \frac{\sqrt{v}}{\sqrt{u}}p$$

Substituting in equation (1), we get

$$\sqrt{v}\left(\frac{u}{v}\right)\left(\frac{dv}{du}\right)^2 + u^{3/2}\left(\frac{\sqrt{u}}{\sqrt{v}}\right)\left(\frac{dv}{du}\right) - u\sqrt{v} = 0$$

or
$$\frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2 + \frac{u^2}{\sqrt{v}}\frac{dv}{du} - u\sqrt{v}$$

or
$$\left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} - v = 0$$

Letting $\frac{dv}{du} = P$, then the above equation becomes

$$P^2 + uP - v = 0$$

or
$$v = uP + P^2$$

which is of Clairaut equation. Thus its general solution is

$$v = uC + C^2$$

or
$$y^2 = x^2 C + C^2$$

**EXAMPLE (7):** **Solve the differential equation :**

$$y = 3xp + 6y^2 p^2$$

**SOLUTION:** Multiply the given equation by $y^2$, we get

$$y^3 = 3xy^2 p + 6y^4 p^2 \qquad (1)$$

Using the transformation $y^3 = u$, then $3y^2 \frac{dy}{dx} = \frac{dx}{dx}$

or $\quad 3y^2 p = \frac{du}{dx} \qquad \left(\text{since } \frac{dy}{dx} = p\right)$

Thus equation (1) becomes

$$u = x\frac{du}{dx} + 6\left(\frac{1}{9}\right)\left(\frac{du}{dx}\right)^2$$

$$= x\frac{du}{dx} + \frac{2}{3}\left(\frac{du}{dx}\right)^2$$

which is of Clairaut form . Thus the solution is $\quad u = Cx + \frac{2}{3}C^2$

or $\quad y^3 = Cx + \frac{2}{3}C^2$

## CLAIRAUT EQUATION

**PROBLEM (4):** **Solve the following differential equations :**

**(i)** $\sin px \cos y = \cos px \sin y + p$

**(ii)** $(y - px)^2 = 1 + p^2$

**(iii)** $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

**(iv)** $p^2 x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0$

**SOLUTION:** (i) $\sin px \cos y = \cos px \sin y + p$

or $\sin px \cos y - \cos px \sin y = p$

or $\sin(px - y) = p$

or $px - y = \sin^{-1} p$

or $y = px - \sin^{-1} p$

which is of Clairaut equation . Hence its general solution is

$$y = Cx - \sin^{-1} C$$

(ii) $(y-px)^2 = 1+p^2$

This equation can be written as

$$y = px \pm \sqrt{1+p^2}$$

Both the component equations are of Clairaut form. Thus the general solution of this equation is

$$y = Cx \pm \sqrt{1+C^2}$$

or $(y-Cx)^2 = 1+C^2$

(iii) $p^2(x^2-a^2)-2pxy+y^2-b^2 = 0$

or $y^2-2pxy+p^2x^2 = a^2p^2+b^2$

or $(y-px)^2 = a^2p^2+b^2$

or $y-px = \pm\sqrt{a^2p^2+b^2}$

or $y = px \pm \sqrt{a^2p^2+b^2}$

Both the components equations are of Clairaut form. Thus the general solution of this equation is

$$y = Cx \pm \sqrt{a^2c^2+b^2}$$

or $(y-Cx)^2 = a^2c^2+b^2$

(iv) $p^2x(x-2)+p(2y-2xy-x+2)+y^2+y = 0$

or $(y^2-2pxy+p^2x^2)+2p(y-px)+(y-px)+2p = 0$

or $(y-px)^2+(2p+1)(y-px)+2p = 0$

or $(y-px+2p)(y-px+1) = 0$

Both the component equations are of Clairaut form. Thus the general solution is

$$(y-Cx+2C)(y-Cx+1) = 0$$

## EQUATIONS REDUCIBLE TO CLAIRAUT FORM

**PROBLEM (5):** **Solve the differential equation :**

$$x^2(y-px) = yp^2$$

**by reducing it to Clairaut form.**

**SOLUTION:** $x^2(y-px) = yp^2$ (1)

Multiplying equation (1) by $y$ and re–arranging

$$x^2y^2-px^3y = y^2p^2$$

or $y^2 = pxy+\left(\frac{yp}{x}\right)^2$

which is of the form $y^2 = pxy+f\left(\frac{y.p}{x}\right)$

Let $x^2 = u$ and $y^2 = v$, then $2x\,dx = du$ and $2y\,dy = dv$

so that $\frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\frac{dy}{dx} = \frac{\sqrt{v}}{\sqrt{u}}p$

Substituting in equation (1), we get

$$u\left(\sqrt{v}-\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{u}\right)=\sqrt{v}\frac{u}{v}\left(\frac{dv}{du}\right)^2$$

or $$u\sqrt{v}-\frac{u^2}{\sqrt{v}}\frac{dv}{du}=\frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2$$

or $$v-u\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$$

or $$v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$$

Letting $\frac{dv}{du}=P$, then the above equation becomes

$$v=uP+P^2$$

which is of Clairaut form. Thus the solution is given by

$$v=uC+C^2$$

or $$y^2=Cx^2+C^2$$

**PROBLEM (6):** **Solve the differential equation :**

$$(px-y)(py+x)=2p$$

**SOLUTION:** Let $x^2=u$ and $y^2=v$, then $2x\,dx=du$ and $2y\,dy=dv$

or $$\frac{dv}{du}=\frac{y}{x}\frac{dy}{dx}=\frac{\sqrt{v}}{\sqrt{u}}p$$

Substituting in the given equation, we get

$$\left(\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{u}-\sqrt{v}\right)\left(\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{v}+\sqrt{u}\right)=2\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$$

$$\left(u\frac{dv}{du}-v\right)\left(\sqrt{u}\frac{dv}{du}+\sqrt{u}\right)=2\sqrt{u}\frac{dv}{du}$$

or $$\left(u\frac{dv}{du}-v\right)\left(\frac{dv}{dx}+1\right)=2\frac{dv}{dx}$$

or $$u\left(\frac{dv}{du}\right)^2+(u-v-2)\frac{dv}{dx}-v=0$$

Letting $\frac{dv}{du}=P$, then

$$uP^2+(u-v-2)P-v=0$$

or $$v=uP^2+(u-v-2)P$$

or $$(1+P)v=uP^2+uP-2P=uP(1+P)-2P$$

or $$v=uP-\frac{2P}{1+P}$$

which is of Clairaut form and has the solution

$$v = uC - \frac{2C}{1+C}.$$

Replacing $u$ by $x^2$ and $v$ by $y^2$, we get

$$y^2 = Cx^2 - \frac{2C}{1+C}$$

as the required solution.

**PROBLEM (7):** **Solve the following differential equation :**

$$\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0$$

**by reducing it to Clairaut form using the transformation $\sin x = u$, $\sin y = v$.**

**SOLUTION:**

$$\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0 \qquad (1)$$

Since $\sin x = u$, $\sin y = v$, therefore

$$\cos x\, dx = du \quad \text{and} \quad \cos y\, dy = dv$$

Then $\frac{dv}{du} = \frac{\cos y\, dy}{\cos x\, dx} = \frac{\cos y}{\cos x}\frac{dy}{dx} = \frac{\cos y}{\cos x}p$

Thus equation (1) becomes

$$\cos^2 y \frac{\cos^2 x}{\cos^2 y}\left(\frac{dv}{du}\right)^2 + u \cos x \cos y \frac{\cos x}{\cos y}\frac{dv}{du} - v\cos^2 x = 0$$

$$\cos^2 x\left(\frac{dv}{du}\right)^2 + u\cos^2 x\left(\frac{dv}{du}\right) - v\cos^2 x = 0$$

or $\left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} - v = 0$

or $v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$

Letting $\frac{dv}{du} = P$, the above equation becomes

$$v = uP + P^2$$

which is of Clairaut form. The general solution is given by

$$v = uC + C^2$$

or $\sin y = C\sin x + C^2$

**PROBLEM (8):** **Solve the following differential equation :**

$$e^{4x}(p-1) + e^{2y}p^2 = 0$$

**by reducing it to Clairaut form using the transformation**

**$e^{2x} = u$ and $e^{2y} = v$.**

**SOLUTION:** $e^{4x}(p-1)+e^{2y}p^2 = 0$ (1)

Since $e^{2x} = u$ and $e^{2y} = v$, therefore

$$2e^{2x}\,dx = du \quad\text{and}\quad 2e^{2y}\,dy = dv$$

Then $\dfrac{dv}{du} = \dfrac{e^{2y}}{2e^{2x}}\dfrac{dy}{dx} = \dfrac{v}{u}p$

Substituting these in equation (1), we get

$$u^2\left(\frac{u}{v}\frac{dv}{du}-1\right)+v\frac{u^2}{v^2}\left(\frac{dv}{du}\right)^2 = 0$$

or $$\frac{u^3}{v}\frac{dv}{du}-u^2+\frac{u^2}{v}\left(\frac{dv}{du}\right)^2 = 0.$$

$$\frac{u}{v}\frac{dv}{du}-1+\frac{1}{v}\left(\frac{dv}{du}\right)^2 = 0$$

or $$u\frac{dv}{du}-v+\left(\frac{dv}{du}\right)^2 = 0$$

or $$v = u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$$

Letting $\dfrac{dv}{du} = P$, the above equation becomes

$$v = uP+P^2$$

which is of Clairaut form. The general solution is given by

$$v = uC+C^2$$

or $$e^{2y} = Ce^{2x}+C^2$$