Before we get into surface integrals we first need to talk about how to parameterize a surface. When we parameterized a curve we took values of \( t \) from some interval \([a, b]\) and plugged them into
\[
\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}
\]
and the resulting set of vectors will be the position vectors for the points on the curve.

With surfaces we'll do something similar. We will take points, \((u, v)\), out of some two-dimensional space \(D\) and plug them into
\[
\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}
\]
and the resulting set of vectors will be the position vectors for the points on the surface \(S\) that we are trying to parameterize. This is often called the \textbf{parametric representation} of the \textbf{parametric surface} \(S\).

We will sometimes need to write the \textbf{parametric equations} for a surface. There are really nothing more than the components of the parametric representation explicitly written down.
\[
\begin{align*}
x &= x(u, v) \\
y &= y(u, v) \\
z &= z(u, v)
\end{align*}
\]

### Example 1
Determine the surface given by the parametric representation
\[
\vec{r}(u, v) = u\hat{i} + u\cos v\hat{j} + u\sin v\hat{k}
\]

**Solution**
Let's first write down the parametric equations.
\[
\begin{align*}
x &= u \\
y &= u\cos v \\
z &= u\sin v
\end{align*}
\]

Now if we square \(y\) and \(z\) and then add them together we get,
\[
y^2 + z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2\left(\cos^2 v + \sin^2 v\right) = u^2 = x^2
\]

So, we were able to eliminate the parameters and the equation in \(x, y,\) and \(z\) is given by,
\[
x^2 = y^2 + z^2
\]

From the \textbf{Quadric Surfaces} section notes we can see that this is a cone that opens along the \(x\)-axis.

We are much more likely to need to be able to write down the parametric equations of a surface than identify the surface from the parametric representation so let's take a look at some examples of this.

### Example 2
Give parametric representations for each of the following surfaces.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>(a) The elliptic paraboloid (x = 5y^2 + 2z^2 - 10). [Solution]</td>
<td>(b) The elliptic paraboloid (x = 5y^2 + 2z^2 - 10) that is in front of the (yz)-plane. [Solution]</td>
</tr>
<tr>
<td>(c) The sphere (x^2 + y^2 + z^2 = 36). [Solution]</td>
<td>(d) The cylinder (y^2 + z^2 = 25). [Solution]</td>
</tr>
</tbody>
</table>
Solution

(a) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$.

This one is probably the easiest one of the four to see how to do. Since the surface is in the form $x = f(y, z)$ we can quickly write down a set of parametric equations as follows,

\[ x = 5y^2 + 2z^2 - 10 \quad y = y \quad z = z \]

The last two equations are just there to acknowledge that we can choose $y$ and $z$ to be anything we want them to be. The parametric representation is then,

\[ \vec{r}(y, z) = \left( 5y^2 + 2z^2 - 10 \right) \hat{i} + y \hat{j} + z \hat{k} \]

(b) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$ that is in front of the $yz$-plane.

This is really a restriction on the previous parametric representation. The parametric representation stays the same.

\[ \vec{r}(y, z) = \left( 5y^2 + 2z^2 - 10 \right) \hat{i} + y \hat{j} + z \hat{k} \]

However, since we only want the surface that lies in front of the $yz$-plane we also need to require that $x \geq 0$. This is equivalent to requiring,

\[ 5y^2 + 2z^2 - 10 \geq 0 \quad \text{or} \quad 5y^2 + 2z^2 \geq 10 \]

(c) The sphere $x^2 + y^2 + z^2 = 36$.

This one can be a little tricky until you see how to do it. In spherical coordinates we know that the equation of a sphere of radius $a$ is given by,

\[ \rho = a \]

and so the equation of this sphere (in spherical coordinates) is $\rho = 3\sqrt{3}$. Now, we also have the following conversion formulas for converting Cartesian coordinates into spherical coordinates.

\[ x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi \]

However, we know what $\rho$ is for our sphere and so if we plug this into these conversion formulas we will arrive at a parametric representation for the sphere. Therefore, the parametric representation is,

\[ \vec{r}(\theta, \varphi) = 3\sqrt{3} \sin \varphi \cos \theta \hat{i} + 3\sqrt{3} \sin \varphi \sin \theta \hat{j} + 3\sqrt{3} \cos \varphi \hat{k} \]

All we need to do now is come up with some restriction on the variables. First we know that we have the following restriction.

\[ 0 \leq \varphi \leq \pi \]

This is enforced upon us by choosing to use spherical coordinates. Also, to make sure that we only trace out the sphere once we will also have the following restriction.

\[ 0 \leq \theta \leq 2\pi \]

(d) The cylinder $y^2 + z^2 = 25$.

As with the last one this can be tricky until you see how to do it. In this case it makes some sense to use cylindrical coordinates since they can be easily used to write down the equation of a cylinder.
In cylindrical coordinates the equation of a cylinder of radius $a$ is given by 

$$r = a$$

and so the equation of the cylinder in this problem is $r = 5$.

Next, we have the following conversion formulas.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Notice that they are slightly different from those that we are used to seeing. We needed to change them up here since the cylinder was centered upon the $x$-axis.

Finally, we know what $r$ is so we can easily write down a parametric representation for this cylinder.

$$\vec{r}(x, \theta) = x \hat{i} + 5 \sin \theta \hat{j} + 5 \cos \theta \hat{k}$$

We will also need the restriction $0 \leq \theta \leq 2\pi$ to make sure that we don’t retrace any portion of the cylinder. Since we haven’t put any restrictions on the “height” of the cylinder there won’t be any restriction on $x$.

In the first part of this example we used the fact that the function was in the form $x = f(y, z)$ to quickly write down a parametric representation. This can always be done for functions that are in this basic form.

$$z = f(x, y) \quad \Rightarrow \quad \vec{r}(x, y) = x \hat{i} + y \hat{j} + f(x, y) \hat{k}$$

$$x = f(y, z) \quad \Rightarrow \quad \vec{r}(y, z) = f(y, z) \hat{i} + y \hat{j} + z \hat{k}$$

$$y = f(x, z) \quad \Rightarrow \quad \vec{r}(x, z) = x \hat{i} + f(x, z) \hat{j} + z \hat{k}$$

Okay, now that we have practice writing down some parametric representations for some surfaces let’s take a quick look at a couple of applications.

Let’s take a look at finding the tangent plane to the parametric surface $S$ given by,

$$\vec{r}(u, v) = x(u, v) \hat{i} + y(u, v) \hat{j} + z(u, v) \hat{k}$$

First, define

$$\vec{r_u}(u, v) = \frac{\partial x}{\partial u}(u, v) \hat{i} + \frac{\partial y}{\partial u}(u, v) \hat{j} + \frac{\partial z}{\partial u}(u, v) \hat{k}$$

$$\vec{r_v}(u, v) = \frac{\partial x}{\partial v}(u, v) \hat{i} + \frac{\partial y}{\partial v}(u, v) \hat{j} + \frac{\partial z}{\partial v}(u, v) \hat{k}$$

Now, provided $\vec{r_u} \times \vec{r_v} \neq \vec{0}$ it can be shown that the vector $\vec{r_u} \times \vec{r_v}$ will be orthogonal to the surface $S$. This means that it can be used for the normal vector that we need in order to write down the equation of a tangent plane. This is an important idea that will be used many times throughout the next couple of sections.

Let’s take a look at an example.

**Example 3** Find the equation of the tangent plane to the surface given by

$$\vec{r}(u, v) = u \hat{i} + 2v \hat{j} + (u^2 + v) \hat{k}$$

at the point $(2, 2, 3)$.
Solution

Let’s first compute \( \vec{r}_u \times \vec{r}_v \). Here are the two individual vectors.

\[
\vec{r}_u (u, v) = \vec{i} + 2u \vec{k} \quad \quad \quad \quad \quad \vec{r}_v (u, v) = 4v \vec{j} + \vec{k}
\]

Now the cross product (which will give us the normal vector \( \vec{n} \)) is,

\[
\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix} = -8uv \vec{i} - 4v \vec{j} + 8u \vec{k}
\]

Now, this is all fine, but in order to use it we will need to determine the value of \( u \) and \( v \) that will give us the point in question. We can easily do this by setting the individual components of the parametric representation equal to the coordinates of the point in question. Doing this gives,

\[
\begin{align*}
2 &= u \\
2 &= 2v^2 \\
3 &= u^2 + v
\end{align*}
\]

Now, as shown, we have the value of \( u \), but there are two possible values of \( v \). To determine the correct value of \( v \) let’s plug \( u \) into the third equation and solve for \( v \). This should tell us what the correct value is.

\[
3 = 4 + v \quad \Rightarrow \quad v = -1
\]

Okay so now we know that we’ll be at the point in question when \( u = 2 \) and \( v = -1 \). At this point the normal vector is,

\[
\vec{n} = 16 \vec{i} - 4 \vec{j} - 4 \vec{k}
\]

The tangent plane is then,

\[
16(x - 2) - (y - 2) - 4(z - 3) = 0 \\
16x - y - 4z = 18
\]

You do remember how to write down the equation of a plane, right?

The second application that we want to take a quick look at is the surface area of the parametric surface \( S \) given by,

\[
\vec{r}(u, v) = x(u, v) \vec{i} + y(u, v) \vec{j} + z(u, v) \vec{k}
\]

and as we will see it again comes down to needing the vector \( \vec{r}_u \times \vec{r}_v \).

So, provided \( S \) is traced out exactly once as \( (u, v) \) ranges over the points in \( D \) the surface area of \( S \) is given by,

\[
A = \iint_D \| \vec{r}_u \times \vec{r}_v \| \, dA
\]

Let’s take a look at an example.

Example 4 Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder \( x^2 + y^2 = 12 \) and above the \( xy \)-plane.
Solution

Okay we’ve got a couple of things to do here. First we need the parameterization of the sphere. We parameterized a sphere earlier in this section so there isn’t too much to do at this point. Here is the parameterization.

\[ \vec{r}(\theta, \varphi) = 4 \sin \varphi \cos \theta \hat{i} + 4 \sin \varphi \sin \theta \hat{j} + 4 \cos \varphi \hat{k} \]

Next we need to determine \( D \). Since we are not restricting how far around the \( z \)-axis we are rotating with the sphere we can take the following range for \( \theta \).

\[ 0 \leq \theta \leq 2\pi \]

Now, we need to determine a range for \( \varphi \). This will take a little work, although it’s not too bad. First, let’s start with the equation of the sphere.

\[ x^2 + y^2 + z^2 = 16 \]

Now, if we substitute the equation for the cylinder into this equation we can find the value of \( z \) where the sphere and the cylinder intersect.

\[ \begin{align*}
  x^2 + y^2 + z^2 &= 16 \\
  12 + z^2 &= 16 \\
  z^2 &= 4 \\
  \Rightarrow \quad z &= 2 
\end{align*} \]

Now, since we also specified that we only want the portion of the sphere that lies above the \( xy \)-plane we know that we need \( z = 2 \). We also know that \( \rho = 4 \). Plugging this into the following conversion formula we get,

\[ z = \rho \cos \varphi \]

\[ 2 = 4 \cos \varphi \]

\[ \cos \varphi = \frac{1}{2} \quad \Rightarrow \quad \varphi = \frac{\pi}{3} \]

So, it looks like the range of \( \varphi \) will be,

\[ 0 \leq \varphi \leq \frac{\pi}{3} \]

Finally, we need to determine \( \vec{r}_\theta \times \vec{r}_\varphi \). Here are the two individual vectors.

\[ \begin{align*}
  \vec{r}_\theta(\theta, \varphi) &= -4 \sin \varphi \sin \theta \hat{i} + 4 \sin \varphi \cos \theta \hat{j} \\
  \vec{r}_\varphi(\theta, \varphi) &= 4 \cos \varphi \cos \theta \hat{i} + 4 \cos \varphi \sin \theta \hat{j} - 4 \sin \varphi \hat{k} 
\end{align*} \]

Now let’s take the cross product.

\[ \vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  -4 \sin \varphi \sin \theta & 4 \sin \varphi \cos \theta & 0 \\
  4 \cos \varphi \cos \theta & 4 \cos \varphi \sin \theta & -4 \sin \varphi 
\end{vmatrix} \]

\[ = -16 \sin^2 \varphi \cos \theta \hat{i} - 16 \sin \varphi \cos \varphi \sin^2 \theta \hat{k} - 16 \sin^2 \varphi \sin \theta \hat{j} - 16 \sin \varphi \cos \varphi \cos^2 \theta \hat{k} \]

We now need the magnitude of this,
\[ \left\| \vec{r}_\phi \times \vec{r}_\theta \right\| = \sqrt{256\sin^4 \varphi \cos^2 \theta + 256\sin^4 \varphi \sin^2 \theta + 256 \sin^2 \varphi \cos^2 \varphi} \\
= \sqrt{256\sin^2 \varphi \left( \cos^2 \theta + \sin^2 \theta \right) + 256 \sin^2 \varphi \cos^2 \varphi} \\
= \sqrt{256 \sin^2 \varphi \left( \sin^2 \varphi + \cos^2 \varphi \right)} \\
= 16 \sin \varphi \\
= 16 \sin \varphi \]

We can drop the absolute value bars in the sine because sine is positive in the range of \( \varphi \) that we are working with.

We can finally get the surface area.

\[
A = \iint_D 16 \sin \varphi \, dA \\
\quad = \int_0^{2\pi} \int_0^\pi 16 \sin \varphi \, d\varphi \, d\theta \\
\quad = \int_0^{2\pi} -16 \cos \varphi \bigg|_0^\pi \, d\theta \\
\quad = \int_0^{2\pi} 8 \, d\theta \\
\quad = 16\pi
\]